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CONTRIBUTIONS TO THE
THEORY OF RANK TESTS

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
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by

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FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Contributions to the Theory of Rank Tests", submitted by J. Sarangi in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The purpose of this study is to investigate the asymptotic behaviour of certain (conditional) rank tests for comparative experiments proposed by Hodges and Lehmann (1962) [8], and their extensions and variations. The results obtained in this thesis are extensions of the results of Mehra and Sarangi (1965) [14] (where the case of complete designs was discussed) to the case of balanced and partially balanced incomplete block designs. The methods employed are based on certain properties of the U-statistics derived by Hoeffding (1948) [9].

In Chapter I, a brief resume of the relevant earlier work has been given and the test proposed by Hodges and Lehmann described. In Chapter II, a simplified large sample version of this test is discussed, which reduces its application to reading the critical value from the chi-square tables.

In Chapter III, the asymptotic distribution of the test statistic, under a sequence of alternatives approaching the null hypothesis has been found to follow a non-central chi-square law. This helps us to study the power properties and the Pitman efficiency of the test statistics in Chapter IV. Explicit numerical evaluations of the expressions for asymptotic efficiency, which has been limited to the normal form of the parent distribution, indicates a rather high degree of efficiency of the test.

Chapters IV and V contain several remarks that follow from the results of our investigation, including a useful lower bound for the asymptotic efficiency of the investigated test relative to the classical F-test. Further investigations that will throw more light on our present findings have been suggested at suitable places.

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CHAPTER I

THE PROBLEM AND THE DESCRIPTION OF THE TEST PROCEDURES

1.1. Introduction

The emphasis on many of the classical test procedures has considerably diminished in the last decade, in relation to a certain class of statistical hypotheses, with attempts to find distribution-free tests. Almost all classical tests are based on the underlying model of "random sampling from a normal population." These test procedures are satisfactory for most practical purposes if the sample is "large" and the sampling distribution of the test statistic can be regarded as approximately normal. But it has now been realised that some of these tests are vulnerable to gross errors if conditions of normality are violated. Their levels of significance are also not exact, unless the parent population is normal.

Non-parametric methods, which overcame the above defects were introduced quite early in statistical literature. But no systematic study of this branch was made till the 1940's. In the last twenty years, this area has been subjected to intensive study and many new non-parametric tests have been proposed and their properties investigated. These tests have invariably been shown to be preferable to the corresponding classical tests.

The scope of our present study is confined to the investigation of the asymptotic behaviour of certain type of rank tests for comparative experiments, proposed by Hodges and Lehmann [8] and their extensions and variations. Subsequently, Lehmann developed a new class of non-

parametric tests, whose theory runs parallel to that of normal theory testing, estimation and multiple comparison procedures. However, the efficiency of their earlier conditional rank test has remained uninvestigated.

Suppose we have k objects or "treatments" (e.g. varieties of wheat, agricultural fertilizers, products from similar manufacturing units etc.) to be compared. In many practical situations our decisions have often to be based on experiments conducted on certain "experimental units." However, in general, different experimental units may react differently towards the same treatment and thus any method of precise comparison should divide the experimental units into relatively homogeneous groups called "blocks." Then the experimental layout can be represented as

TREAT- MENTS BLOCKS	1	2	..	j	..k
1					
2					
:					
i				m_{ij}	
:					
n					

where there are m_{ij} units in the i^{th} block corresponding to the j^{th} treatment (or in the $(i,j)^{\text{th}}$ cell). In what follows, asymptotic means "as n tends to infinity."

(i) ONE-WAY LAYOUT

If the reactions of the experimental units in different blocks towards the same treatment are not different, then our analysis reduces to that of the one-way layout or the k-sample problem. This problem, in the special case when $k = 2$, was considered by Wilcoxon [20] in 1945. In 1948, Pitman [16], who developed the concept of efficiency, found that the efficiency of this test relative to the corresponding normal theory test (t-test) is $(3/\pi)$ if the parent population is normal. In 1952, Kruskal-Wallis [10] generalised Wilcoxon's test to any positive integral k ; and the corresponding asymptotic efficiency was also shown to be $(3/\pi)$ (Andrews [1], 1954).

(ii) TWO-WAY LAYOUT: SEPARATE RANKING

However, if the reactions of the experimental units differ from block to block, the design reduces to a two-way layout. The earliest non-parametric test for this was suggested by Wilcoxon in 1946 for the paired comparison case when $k = 2$ and all $m_{ij} = 1$. Pitman (1948) found the asymptotic efficiency of this test relative to the t-test to be $3/\pi$, if the parent population is normal. Friedman [5], in 1937, proposed a test based on independent rankings within each block for the case when all $m_{ij} = 1$ i.e. there is one observation per cell. Durbin [4] in 1951, extended the test to cover balanced incomplete block designs when $m_{ij} = 0$ or 1 . The asymptotic efficiency of these tests was computed by van-Elteren and Noether (1959, [15]) to be $\{3m/\pi(m+1)\}$, where m is the block size. In 1953, Benard and van-Elteren [2] discussed these tests in great detail and proposed tests for a wide class of incomplete block designs. Hodges-Lehmann [7] showed that the asymptotic efficiency of Wilcoxon test relative to the t-test never falls below .864, although its

value can be arbitrarily high and even infinite. This very desirable feature of attaining a high efficiency independently of what the parent population is, has been called "robustness" in statistical literature.

TWO-WAY LAYOUT: JOINT RANKING

It is noted that for $k = 2$ and all $m_{ij} = 1$, the paired comparison test attains an asymptotic efficiency of $3/\pi$ relative to the t -test against $2/\pi$ of Friedman's test. The higher efficiency of the former seems to be due to the fact that it takes account of certain interblock comparisons which are entirely ignored by the latter. A first attempt at the investigation of a test statistic based on interblock comparisons was done by Mehra [13] for $m = 2$ and general k ; and the corresponding asymptotic efficiencies agreed with that of the paired-comparison test relative to the competing test procedures. Mehra also extended the results to certain paired comparison experimental designs. Hodges and Lehmann [8], in 1962, also proposed a conditional test based on a combined ranking of all the observations which thus takes into account some interblock comparisons of cell observations. In this chapter, we discuss an asymptotic version of the proposed test. The asymptotic efficiency of the case of complete experimental designs was discussed by Mehra and Sarangi [14]. Here the results of Hodges-Lehmann [8] and Mehra-Sarangi [14] are extended to other general designs like the balanced and partially balanced incomplete block designs etc.

1.2. Notations

Let X_{ijl} ($i = 1, 2, \dots, n; j = 1, 2, \dots, k; l = 1, 2, \dots, m_{ij}$) be the l^{th} observation in the (i, j) th. cell. We assume these m_{ij} independent random variables (r.v.) to be identically distributed with a common continuous

cumulative distribution function (c.d.f.) $F_{ij}(x)$ such that

$$(1.2.1) \quad \Omega : F_{ij}(x) = F_j(x + \xi_i)$$

where ξ_i 's are unknown constants called "block effects." The null hypothesis of no treatment differences is equivalent to

$$H_0 : F_1 = F_2 = \dots = F_k$$

We note that model (1.2.1) is the assumption of additivity of block effects in the classical linear analysis of variance model.

1.3. Method of Alignment

The conditional test proposed by Hodges-Lehmann [8] is based on a combined ranking of all observations in different blocks after they have been made comparable by removing the block effects. The removal of the block effects is referred to as "alignment." One method of alignment is to subtract from each observation in a block, say the i^{th} , a function μ of the observations in that block, satisfying the condition.

$$(1.3.1) \quad \mu(x_{i11} + a, x_{i12} + a, \dots, x_{i1m_{i1}} + a, \dots, x_{ik1} + a, \dots, x_{ikm_{ik}} + a) \\ = \mu(x_{i11}, x_{i12}, \dots, x_{i1m_{i1}}, \dots, x_{ik1}, \dots, x_{ikm_{ik}}) + a$$

It may be observed that the aligned observations in each block should preferably have a joint symmetric distribution in its arguments under the hypothesis. This condition is not necessary e.g. our method of alignment may be such that the joint distribution is symmetric in disjoint subsets of its arguments. But we confine our attention only to the former case, which is satisfied if, in particular, μ is a symmetric function of its arguments. This is shown by the following lemma.

LEMMA. 1.1. Let $X_1, X_2, \dots, X_c$ be independent identically distributed r.v.'s and let $\mu(X_1, X_2, \dots, X_c)$ be a symmetric function of these variables. Then, the r.v.'s $Z_1, Z_2, \dots, Z_c$ where $Z_j = X_j - \mu(X_1, X_2, \dots, X_c)$ have a symmetric

joint distribution.

Proof: This lemma was proved in [14]. We also give the proof below.

$$\begin{aligned}
 F(t_2, t_1, \dots, t_c) &= \Pr[Z_1 \leq t_2, Z_2 \leq t_1, \dots, Z_c \leq t_c] \\
 &= \Pr[X_1 - \mu(X_1, X_2, \dots, X_c) \leq t_2, X_2 - \mu(X_1, X_2, \dots, X_c) \\
 &\quad \leq t_1, Z_3 \leq t_3, \dots, Z_c \leq t_c] \\
 &= \Pr[X_2 - \mu(X_2, X_1, \dots, X_c) \leq t_2, X_1 - \mu(X_2, X_1, \dots, X_c) \\
 &\quad \leq t_1, \dots, Z_c \leq t_c] \\
 &= \Pr[Z_2 \leq t_2, Z_1 \leq t_1, \dots, Z_c \leq t_c] \\
 &= F(t_1, t_2, \dots, t_c)
 \end{aligned}$$

by the symmetry of $\mu(X_1, X_2, \dots, X_c)$.

It is seen that statistics like the arithmetic mean, median etc. satisfy the above conditions.

1.4. Description of the test

A fairly complete discussion of the test and its asymptotic versions are given in [8] for the special case of $k = 2$. Our discussion concerns the asymptotic version of the test for general k . Let Z_{ijl} ($i = 1, 2, \dots, n; j = 1, 2, \dots, k; l = 1, 2, \dots, m_{ij}$) denote the aligned observations and r_{ijl} denote the rank of Z_{ijl} in a combined ranking of all the $N = \sum_{i,j} m_{ij}$ aligned observations. Following [8], each conditional event when the set of ranks in each block is given will be referred to as a "configuration." Given a configuration, under the null hypothesis of no treatment differences, the ranks within each block can be assigned to the treatments at random, since the vector of aligned observations within each block has a symmetric joint distribution. The following discussion assumes (i) that the method of alignment is the same in each block and (ii) that conditions (A) and (B) of [8], given below are satisfied.

(A) After alignment, each block contains at least one positive and one negative observation.

(B) Complete randomization is used within each block.

These conditions are very general and satisfied for most methods of alignment like those on the mean, median or the Winsorized and censored means. Let $N_i = \sum_j m_{ij}$, $R_j = \sum_i r_{ij}$ = sum of the ranks under the j^{th} treatment. Following [14], let $\tilde{E}(\cdot)$, $\tilde{\text{var}}(\cdot)$ and $\tilde{\text{cov}}(\cdot, \cdot)$ denote the conditional expectation, variance and covariance respectively, under H_0 , given a configuration. Then by direct computation, we get

$$(1.4.1) \quad \left\{ \begin{array}{l} \tilde{E}(R_j) = \sum_{i=1}^n m_{ij} \bar{r}_i \\ \sigma_j^2 = \sigma_{jj} = \tilde{\text{var}}(R_j) = \sum_{i=1}^n (N_i - m_{ij}) m_{ij} \{T_i^2 / (N_i - 1)\} \\ \sigma_{jj'} = \tilde{\text{cov}}(R_j, R_{j'}) = - \sum_{i=1}^n m_{ij} m_{ij'} \{T_i^2 / (N_i - 1)\} \end{array} \right.$$

where $T_i^2 = \sum_{j,l} N_i^{-1} (r_{ijl} - \bar{r}_i)^2$ and $\bar{r}_i = (\sum_l r_{ijl} / N_i)$. We now prove:

Theorem. 1.1.

If (c') every pair of treatments occurs together at least in one block, then the variance-covariance matrix has rank (k-1).

Proof: The theorem is a direct consequence of the following result due to Taussky [18].

Theorem (A). If $(\sigma_{jj'})$ be a $k \times k$ matrix of complex elements such that

$$|\sigma_{jj}| > \sum_{j' \neq j=1}^k |\sigma_{jj'}| \quad (j = 1, 2, \dots, k)$$

then $|(\sigma_{jj'})| \neq 0$.

In our case, $\sigma_{jj} > 0$, $\sigma_{jj'} < 0$ and $|\sigma_{jj}| = \sum_{j' \neq j=1}^k |\sigma_{jj'}|$. Thus the matrix obtained from $(\sigma_{jj'})$ by deleting the j^{th} row and j^{th} column satisfies

the conditions of the theorem. Hence this matrix and thus the variance-covariance matrix (singular) has rank $(k - 1)$.

The proposed conditional test statistic of [8] can now be obtained as follows. Consider the matrix obtained from

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1k} & \{R_1 - \tilde{E}(R_1)\} \\ \sigma_{12} & \sigma_{22} & \dots & \sigma_{2k} & \{R_2 - \tilde{E}(R_2)\} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{1k} & \sigma_{2k} & \dots & \sigma_{kk} & \{R_k - \tilde{E}(R_k)\} \\ \{R_1 - \tilde{E}(R_1)\} & \{R_2 - \tilde{E}(R_2)\} & \dots & \{R_k - \tilde{E}(R_k)\} & 0 \end{vmatrix}$$

by deleting the j^{th} row and the j^{th} column ($j \neq k+1$) and compute its determinant $|\Delta_u|$; consider the above matrix without the last row and last column and compute the determinant $|\Delta|$ of the matrix obtained by deleting the same row and column from it. The test statistic is

$$(1.4.2) \quad W_n = |\Delta_u| / |\Delta|$$

The test consists in rejecting H_0 at a level of significance α , if W_n exceeds a predetermined number $W_{n,\alpha}$ such that the conditional probability $\tilde{P}_{H_0} [W_n > W_{n,\alpha}] = \alpha$. The application of this test requires the null distribution of W_n which can be done by random allocation of the ranks within each block to the treatments present there. A computing machine may be used unless the design is small for manual operations.

The statistic W_n takes a more convenient form when m_{ij} does not vary from block to block, viz.

$$(1.4.3) \quad m_{1j} = m_{2j} = \dots = m_{nj} = m_j \geq 1, \quad (j = 1, 2, \dots, k)$$

We give here details of this case, which was dealt with in [14]. Under (1.4.3), the expressions (1.4.1) simplify to

$$\widetilde{E}(R_j) = m_j n(nN' + 1)/2$$

$$\sigma_{jj} = \widetilde{\text{Var}}(R_j) = \{(N' - m_j)m_j / (N' - 1)\} \left(\sum_{i=1}^n T_i^2 \right)$$

$$\sigma_{jj'} = \widetilde{\text{Cov}}(R_j, R_{j'}) = - \{m_j m_{j'} / (N' - 1)\} \left(\sum_{i=1}^n T_i^2 \right)$$

where $N' = \sum_{j=1}^k m_j$. Here, the inverse of the variance-covariance matrix, after deleting the last row and last column is easily verified to be $\Delta^{-1} = (e_{jj'})$, where,

$$e_{jj} = \frac{N' - 1}{N' \left(\sum_{i=1}^n T_i^2 \right)} \left(\frac{1}{m_k} + \frac{1}{m_j} \right)$$

$$e_{jj'} = + (N' - 1)/N' m_k \left(\sum_{i=1}^n T_i^2 \right) \quad (j \neq j' = 1, 2, \dots, (k-1))$$

It is easily verified, using the relation $\sum \{R_j - m_j n(nN' + 1)/2\} = 0$, that,

$$(1.4.4) \quad W_n = \left[(N' - 1)/N' \left(\sum_{i=1}^n T_i^2 \right) \right] \left[\sum_{j=1}^k \frac{1}{m_j} \{R_j - m_j n(nN' + 1)/2\}^2 \right]$$

For the simpler case of one observation per cell, which has drawn special attention in statistical literature, we get

$$W_n = \left[(k - 1)/k \left(\sum_{i=1}^n T_i^2 \right) \right] \left[\sum_{j=1}^k \{R_j - n(nk + 1)/2\}^2 \right]$$

CHAPTER II

ASYMPTOTIC VERSION OF THE TEST UNDER THE HYPOTHESIS

2.1. Asymptotic Null Distribution

It has been pointed out in the last chapter that the computation of the null distribution of W_n is rather cumbersome and the magnitude of labour increases rapidly with increase in sample size. However, when the sample size or the number of blocks is large, the critical value $W_{n,\alpha}$ can be approximated by using χ^2 - distribution tables, as shown by the following theorem. It is important to note that in the following theorem, the convergence of W_n to the χ^2 - variable is uniform in the configuration.

Theorem 2.1. Suppose the hypothesis H_0 is true, the method of alignment be such that (A) and (B) hold and also the conditions (C) as n tends to infinity, $\{N_{jj},(n)/n\}$ is bounded away from zero for each pair (j,j') , where $N_{jj},(n)$ denotes the number of blocks in which both the treatments T_j and $T_{j'}$ occur together and (D) $N_i < n$ for all i . Then (i) the conditional test statistic W_n , given a configuration, converges in distribution, as n tends to infinity, to the χ^2 - variable with $(k-1)$ degrees of freedom; and (ii) the convergence is uniform in the configuration.

This theorem was proved by Mehra and Sarangi [14] under the restriction $m_{ij} \geq 1$ i.e. for complete experimental designs. These results are extended below to the case where some m_{ij} 's are zero i.e. incomplete block designs.

Let $R_{ij} = \sum_{l=1}^{m_{ij}} r_{ijl}$. Consider an arbitrary linear combination $U_i = \sum_{j=1}^{k-1} c_j R_{ij}$ of R_{ij} , $j = 1, 2, \dots, (k-1)$. Let b_i^2 and β_i denote the conditional variance and third absolute moment of U_i and let $S_n^2 = \sum_{i=1}^n b_i^2$.

The proof of the theorem follows from the following Lemmas.

Lemma 2.1. Under the assumptions (A), (B), (C) and (D), (i) the conditional (given a configuration) distribution of $\sum_{i=1}^n \{U_i - E(U_i)\}/S_n$ converges, as $n \rightarrow \infty$, to the standard normal distribution; and (ii) the convergence is uniform in the configuration.

Proof. In the sequel, let S_j denote the set of all blocks containing the j^{th} treatment and $S_{jj'}$ denote the set of all blocks containing both the j^{th} and j'^{th} treatments. The following possibilities can occur.

(i) Suppose $c_j \neq 0$ and $c_{j'} = 0$ for all $j' \neq j$. Then, there exists a constant $0 < a_1 < \infty$, independent of the configuration such that

$$(2.1.1) \quad \sum_{i=1}^n \beta_i \leq |c_j|^3 \sum_{i=1}^n \tilde{E} |R_{ij} - \tilde{E} R_{ij}|^3 \leq a_1 n^4$$

Further, from (1.4.1),

$$S_n^2 = c_j^2 \sum_{i=1}^n (N_i - m_{ij}) m_{ij} \{T_i^2 / (N_i - 1)\} \geq c_j^2 \sum_{i \in S_j} T_i^2 / (n-1)$$

By assumption (C), there exists $0 < f \leq 1$, independent of j such that for every $n > n_0$ (to be chosen large enough), there are more than nf elements in S_j . If the original aligned observations consist of s negative and $(N - s)$ non-negative numbers; every positive observation must have a rank greater than s and every negative observation a rank not greater than s . In any case, by (A), for every i there exists a r_{ijl} , such that,

$$|r_{ijl} - \bar{r}_i| > |r_{ijl} - s|$$

Thus,

$$\begin{aligned} \sum_{i \in S_j} T_i^2 &= \sum_{i \in S_j} \sum_{l=1}^N (r_{ijl} - \bar{r}_i)^2 / N_i \geq \sum_{i \in S_j} (r_{ijl} - s)^2 / n \\ &\geq 2 \left[1^2 + 2^2 + \dots + [(nf - 1)/2]^2 \right] / n > a_2 n^3 \end{aligned}$$

where $0 < a_2 < \infty$ is independent of the configuration and $[p]$ denotes the greatest integer contained in p ; since under the continuity of the c.d.f.'s there will be no ties for non-zero aligned observations with probability one. Thus, there exists $0 < a_2 < \infty$, independent of the configuration such that,

$$(2.1.2) \quad S_n^2 > a_2 n^3$$

(ii) Suppose $c_1 = c_2 = \dots = c_{k-1} \neq 0$. Then

$$\sum_{i=1}^n U_i = c_1 [N(N+1)/2 - R_k]$$

since the sum of all the ranks is $N(N+1)/2$. Equations (2.1.1) and (2.1.2) can now be proved as above.

(iii) Suppose there exist at least two c 's, say c_j and $c_{j'}$, which are not equal. Then (2.1.1) follows as before and (2.1.2) is proved below.

$$\begin{aligned} S_n^2 &= \sum_{i=1}^n b_i^2 = \sum_{i=1}^n \left[\sum_{j=1}^{k-1} c_j^2 (N_i - m_{ij}) m_{ij} - 2 \sum_{j < j'} c_j c_{j'} m_{ij} m_{ij'} \right] \\ &\quad \times [T_i^2 / (N_i - 1)] \\ &\geq \sum_{i=1}^n \left[\sum_{j=1}^{k-1} c_j^2 \left\{ m_{ij} / \left(\sum_{j=1}^{k-1} m_{ij} \right) \right\} - \left\{ \left(\sum_{j=1}^{k-1} c_j m_{ij} \right) / \left(\sum_{j=1}^{k-1} m_{ij} \right) \right\}^2 \right] \\ &\quad \times \left[\left(\sum_{j=1}^{k-1} m_{ij} \right)^2 T_i^2 / (N_i - 1) \right] \end{aligned}$$

$$\geq \eta^* \sum_{i \in S_{jj'}} T_i^2 \geq a_2 n^3$$

where $0 < a_2 < \infty$ and $0 < \eta^* < \infty$ is a constant independent of the configuration. The second inequality follows from the fact that the quantity under the first square brackets, being the variance of a random variable which takes values $c_1, c_2, \dots, c_{k-1}$ with probabilities proportional to $m_{i1}, m_{i2}, \dots, m_{i(k-1)}$ is bounded below by a positive constant independent of the configuration and $i \in S_{jj'}$.

Thus the equations (2.1.1) and (2.1.2) hold for an arbitrary linear combination $U_i = \sum_{j=1}^{k-1} c_j R_{ij}$. The rest of Lemma 2.1 follows from Berry-Esseen theorem [3], which can be stated as follows.

Theorem B. Let $U_1, U_2, \dots$ be independently distributed with means $E(U_1), E(U_2), \dots$ and let b_i^2, β_i and S_n^2 have the same meaning as in Lemma 2.1. Let $F_n(x)$ denote the c.d.f. of $\sum_{i=1}^n \{U_i - E(U_i)\}/S_n$, and $\Phi(x)$ the standard normal c.d.f. There exists a constant $c < \infty$ such that for all x ,

$$|F_n(x) - \Phi(x)| \leq c \cdot \sum_{i=1}^n \beta_i / S_n^3$$

The following lemmas have been proved in [14].

Lemma 2.2. (Mehra). Let $V^{(n)} = (V_1^{(n)}, V_2^{(n)}, \dots, V_c^{(n)})$, $n = 1, 2, \dots$ be a sequence of random vectors with c.d.f. $F_\theta^{(n)}(v)$ (in the c -space $R^{(c)}$) depending on a parameter θ , and let $U^{(n)} = f(V^{(n)}) = \sum_{j=1}^c c_j V_j^{(n)}$ ($f: R^{(c)} \rightarrow R^{(1)}$) of the components, such that as $n \rightarrow \infty$, (i) $F_\theta^{(n)}(v)$ converges to $F_\theta(v)$ for each θ (and v), where $F_\theta(v)$ is the c.d.f. of a r.v. V , (ii) for every linear f , $G_\theta^{(n)}(u)$ converges to $G_\theta(u)$ uniformly in θ (and u) where $G_\theta(u)$ is the c.d.f. of the r.v. $f(V)$, then $F_\theta^{(n)}(v)$ also converges to $F_\theta(v)$ uniformly in

θ (and v).

Lemma 2.3. (Mehra). Let the sequences $\{V^{(n)}\}$ and $\{F_{\theta}^{(n)}(v)\}$ $n = 1, 2, \dots$ be as defined in Lemma 2.2 and suppose the conclusion of Lemma 2.2 holds. Let $f : R^{(c)} \rightarrow R^{(1)}$ be a continuous function, then
(i) the distribution $G_{\theta}^{(n)}(u)$ of $f(V^{(n)})$ converges to the distribution $G_{\theta}(u)$ of $f(V)$ and (ii) the convergence is uniform in θ (and u)

Proof of the Theorem 2.1. It follows from part (i) of Lemma 2.1 that the standardized r.v.'s $R_1, R_2, \dots, R_{k-1}$ in the limit, as $n \rightarrow \infty$, have a $(k-1)$ - variate normal distribution. This verifies condition (i) of Lemma 2.2. Part (ii) of Lemma 2.1 verifies condition (ii) of Lemma 2.2. Thus the convergence of the standardized $R_1, R_2, \dots, R_{k-1}$ to the normal distribution is uniform in configuration. Again, the statistic W_n , being a quadratic form in $R_1, R_2, \dots, R_{k-1}$ is a continuous mapping from $R^{(k-1)}$ to $R^{(1)}$. Thus the conditions of Lemma 2.3 hold. The following theorem, proved in Rao [17, p.57] shows that $G_{\theta}(u)$ of Lemma 2.3 is the χ^2 - distribution with $(k-1)$ degrees of freedom. The convergence, by Lemma 2.3 is uniform in configuration.

Theorem C. If the vector $\underline{R} = (R_1 - \tilde{E}(R_1), R_2 - \tilde{E}(R_2), \dots, R_{k-1} - \tilde{E}(R_{k-1}))$ has a normal distribution with mean vector $\underline{\mu}$ and non-singular variance-covariance matrix $\underline{\Delta}$, then the quadratic form $\underline{R}' \underline{\Delta}^{-1} \underline{R}$ has a $\chi^2_{k-1}(\lambda)$ distribution i.e. a non-central chi-square distribution with non-centrality parameter λ and $(k-1)$ degrees of freedom, where $\lambda = \underline{\mu}' \underline{\Delta}^{-1} \underline{\mu}$.

Remark . In proving theorem 2.1 we assumed the aligned observations to be distinct; which is true with probability one if the aligned observations have a continuous c.d.f. In practical applications of the test, equal aligned observations can be allotted distinct ranks after arranging them in some order at random.

An alternative method of ranking will be to allot each member of sets of equal aligned observations the middle rank. For this method of ranking and alignment on the median, the conclusion of theorem 2.1 holds in spite of the fact that many aligned observations may become zero. The proof of Lemma 2.1 goes through so long as there are distinct positive and negative aligned observations in each block.

CHAPTER III

ASYMPTOTIC DISTRIBUTION UNDER THE ALTERNATIVES

3.1. Introduction

In the previous chapter, the asymptotic null distribution of W_n was found under very general conditions on the experimental layout and the method of alignment. In this chapter, we will study the unconditional asymptotic distribution of the statistic W_n for certain sequence of translation alternatives approaching H_0 . For complete experimental designs, with the restriction $m_{ij} = m_j$ for all i (for each j), the limit distribution was studied in Mehra - Sarangi [14]. The results of this section extend the results of [14] to (i) balanced incomplete block designs, when $m_{ij} = 0$ or 1 , and (ii) partially balanced incomplete block designs of a certain type to be specified. We confine our attention to alignment on the mean only. The unconditional distribution of W_n will then enable us to compute the (Pitman) efficiency of this test relative to the F - test, the competing separate ranking tests, and certain tests proposed by Lehmann in [11] and [12].

For every fixed n , consider the alternative hypothesis:

$$(3.1.1) \quad K_n : F_j(x) = F(x + \theta_j / \sqrt{n}) \quad (j = 1, 2, \dots, k)$$

where not all θ_j 's are equal. The method of proof is the same as that of Andrews [1] and depends upon certain properties of U - statistics obtained by Hoeffding [9]. But for the sake of clarity, the two cases mentioned above will be discussed in separate sections.

3.2. Balanced Incomplete Block Designs

Let the experimental layout be a balanced incomplete block design with parameters:

n = number of blocks in the layout.

k = number of treatments to be compared.

m = block size i.e. number of plots in each block.

r = number of times each treatment has been replicated.

ρ = number of blocks in which any two specified treatments occur together.

The following relations are well-known.

$$(3.2.1) \quad n.m = r.k, \quad r(m-1) = \rho(k-1), \quad n \geq k \geq m, \\ r \leq n$$

As before, let S_j denote the set of blocks in which the j^{th} treatment is present, and $S_{jj'}$, the set of blocks in which both the j^{th} and the j'^{th} treatments are present. The sets S_j and $S_{jj'}$ are identified. We may assume (with probability one) that

(E) All aligned observations are different from zero.

The conclusion of the theorems below would hold even without this assumption. We are making such an assumption specifically to overcome notational difficulties.

Let Z_{ij} stand for the aligned observation (under assumption E) in the $(i,j)^{\text{th}}$ cell if $i \in S_j$ and be zero if $i \notin S_j$. Let r_{ij} denote the rank of Z_{ij} in a combined ranking of the totality of aligned observations if $i \in S_j$ and r_{ij} be zero otherwise.

If $X_1, X_2, \dots, X_m$ be m independent random variables distributed identically with c.d.f. $F(x)$, let $G(u)$ denote the c.d.f. of

$Z_1 = (X_1 - \bar{X})$ and let $H(u,v)$ denote the joint c.d.f. of (Z_1, Z_2) where $\bar{X} = (\sum_{i=1}^m X_i / m)$. Let $\bar{\theta}_i$ denote the mean of the θ 's that are present in the i^{th} block. It can be verified by direct computation or from (1.4.1) that for the above-described balanced incomplete block design,

$$(3.2.2) \quad \begin{cases} \widetilde{E}(R_j) = \sum_{i \in S_j} \bar{r}_i = \sum_{j'=1}^k \sum_{i \in S_{jj'}} (r_{ij'} / m) \\ \sigma_{jj} = \widetilde{\text{Var}}(R_j) = \sum_{i \in S_j} (T_i^2 / m - \bar{r}_i^2) \\ \sigma_{jj'} = \widetilde{\text{Cov}}(R_j, R_{j'}) = - \sum_{i \in S_{jj'}} (T_i^2 / m - \bar{r}_i^2) / (m - 1) \end{cases}$$

where $\bar{r}_i = \sum_{j=1}^k r_{ij} / m$ and $T_i^2 = \sum_{j=1}^k r_{ij}^2$

We now prove:

Theorem 3.1. If the hypothesis K_n be true and the following conditions are satisfied (i) $F(x)$ possesses a continuous derivative $F'(x)$ a.e. and (ii) there exists a function $b(x)$ such that $[\{ F(x + \theta) - F(x) \} / \theta] \leq b(x)$, where $\int_{-\infty}^{\infty} b(x) dF(x) < \infty$, and (iii) for every m , there exists a function $b_m(u)$ such that $[\{ G^{(m)}(u + \theta) - G^{(m)}(u) \} / \theta] = [\{ G(u + \theta) - G(u) \} / \theta] \leq b_m(u)$, where $\int_{-\infty}^{\infty} b_m(u) dG(u) < \infty$, then the statistic defined by (1.4.2) converges in distribution, as $n \rightarrow \infty$, to $\chi^2_{k-1}(\Delta_w)$ where,

$$(3.2.3) \quad \Delta_w = \left[3(m-1)/(1-3\lambda)(k-1) \right] \left[\int_{-\infty}^{\infty} \{ G'(u) \}^2 du \right]^2 \sum_{j=1}^k (\theta_j - \bar{\theta})^2$$

with $\bar{\theta} = \sum_{j=1}^k \theta_j / k$ and,

$$\lambda = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(u) G(v) dH(u,v)$$

The proof of the theorem follows from the lemmas proved below.

Let

$$(3.2.4) \quad T_{nj} = \sqrt{3(k-1)} \left\{ R_j - \sum_{j'=1}^k \sum_i r_{ij'} / m \right\} / mn^{3/2} \sqrt{(m-1)(1-3\lambda)}$$

$$(3.2.5) \quad \mu_j = - \sqrt{3(m-1)} \left\{ \int_{-\infty}^{\infty} \{G'(u)\}^2 du \right\} (\theta_j - \bar{\theta}) / \sqrt{(k-1)(1-3\lambda)}$$

Further, let Λ represent the variance-covariance matrix

$$|| \delta_{jj'} - 1/k || \text{ with } j, j' = 1, 2, \dots, k.$$

Lemma 3.1. Under the assumptions of theorem 3.1, the random vector $\underline{T}_n = (T_{n1}, T_{n2}, \dots, T_{nk})$ converges in distribution, as $n \rightarrow \infty$, to the k-variate normal vector with mean vector $\underline{\mu} = (\mu_1, \mu_2, \dots, \mu_k)$ and variance-covariance matrix Λ .

Proof. We may assume without loss of generality that the aligned observations are all different from zero (see assumption E). It is sufficient to prove that the k^2 -statistics given by

$$T_{n(jj')} = \sum_{i \in S_{jj'}} r_{ij} / n^{3/2} \quad (j, j' = 1, 2, \dots, k)$$

have an asymptotic normal distribution; for the asymptotic normality of T_n follows by a linear transformation. The later result is derived here by arguments parallel to those of Andrews [1] and Mehra - Sarangi [14].

Define $\delta(y_j, y_{j'}) = 1$ if $y_j > y_{j'}$; $y_j, y_{j'} \neq 0$ and

$\delta(y_j, y_{j'}) = 0$ otherwise. Further, define $h^j(y_1, y_2, \dots, y_k) = \sum_{j'=1}^k \delta(y_j, y_{j'})$.

Let,

$$(3.2.6) \quad U^{(jj')} = \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_j \in S_{jj'}} \dots \sum_{i_k=1}^n h^j(z_{i_1 1}, z_{i_2 2}, \dots, z_{i_k k}) / n^k$$

It is easily verified that,

$$\begin{aligned}
 \text{The R.H.S.} &= \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_j \in S_{jj'}} \dots \sum_{i_k=1}^n \sum_{j'=1}^k \delta(Z_{i_j j}, Z_{i_{j'} j'}) / n^k \\
 &= \sum_{i_j \in S_{jj'}} \left\{ \sum_{i_{j'}=1}^n \sum_{j'=1}^k \delta(Z_{i_j j}, Z_{i_{j'} j'}) \right\} / n^2 \\
 (3.2.7) \quad &= \left\{ \sum_{i_j \in S_{jj'}} r_{i_j} - \rho \right\} / n^2
 \end{aligned}$$

Define,

$$\begin{aligned}
 \phi^{(jj')} (Z_{i_1}, Z_{i_2}, \dots, Z_{i_k}) &= \sum_{(\alpha_1, \dots, \alpha_k)} h^j(Z_{\alpha_1 1}, Z_{\alpha_2 2}, \dots, Z_{\alpha_k k}) / k! \\
 &\quad \text{if } \{i_1, i_2, \dots, i_k\} \cap S_{jj'} \neq \emptyset \\
 &= 0 \quad \text{otherwise}
 \end{aligned}$$

where $(\alpha_1, \alpha_2, \dots, \alpha_k)$ extends over all possible permutations of the set of integers $\{i_1, i_2, \dots, i_k\}$ such that $\alpha_j \in S_{jj'}$; $\emptyset$ being the empty set. Let,

$$U^{(jj')} = \sum_{(i_1, i_2, \dots, i_k)} \phi^{(jj')} (Z_{i_1}, Z_{i_2}, \dots, Z_{i_k}) / \binom{n}{k}$$

where $(i_1, i_2, \dots, i_k)$ extends over all sets of integers such that $1 \leq i_1, i_2, \dots, i_k \leq n$, and no two of the i 's are equal. $U^{(jj')}$ is the average of $\binom{n}{k} \cdot k!$ terms of the type $h^j(Z_{\alpha_1 1}, Z_{\alpha_2 2}, \dots, Z_{\alpha_k k})$, the zero terms of which however, do not appear in $U^{(jj')}$. It is easily checked that the number of terms of the type $h^j(Z_{\alpha_1 1}, Z_{\alpha_2 2}, \dots, Z_{\alpha_k k})$ which are

averaged to give $U^{(jj')}$ and also occur in $U^{(jj')}$ are $\rho\{(k-1)!\} \binom{n-1}{k-1}$.

Let $J^{(jj')}$ consisting of $\rho[n^{k-1} - \binom{n-1}{k-1}\{(k-1)!\}]$ terms denote the sum of all $h^j(Z_{\alpha_1 1}, Z_{\alpha_2 2}, \dots, Z_{\alpha_k k})$ terms that appear in the expanded form of $U^{(jj')}$ but not in that of $U^{(jj')}$. Then,

$$U^{(jj')} = \left\{ \binom{n}{k} \cdot k! U^{(jj')} + J^{(jj')} \right\} / n^k$$

It can now be verified, as in [14], that

$$\lim_{n \rightarrow \infty} \{ \text{Var } \sqrt{n} (U^{(jj')} - EU^{(jj')}) \} = 0$$

Hence, by the following lemma of Hoeffding, if either $\sqrt{n} (U^{(jj')} - EU^{(jj')})$; $j, j' = 1, 2, \dots, k$ or $\sqrt{n} (U^{(jj')} - EU^{(jj')})$; $j, j' = 1, 2, \dots, k$ has a limiting joint distribution, the other has the same limiting joint distribution.

Lemma D. Let $V_1, V_2, \dots$ be an infinite sequence of random vectors $V_n = (V_n^{(1)}, V_n^{(2)}, \dots, V_n^{(k)})$ and suppose that the c.d.f. $F_n(v)$ of V_n tends to a c.d.f. $F(v)$, as $n \rightarrow \infty$. Let $V_n^{(j)'} = V_n^{(j)} + d_n^{(j)}$, where, $\lim_{n \rightarrow \infty} E\{d_n^{(j)}\}^2 = 0$ for all $j = 1, 2, \dots, k$. Then the distribution function of $V_n' = (V_n^{(1)'}, \dots, V_n^{(k)'})$ tends to $F(v)$. The limiting joint distribution of $\sqrt{n} (U^{(jj')} - EU^{(jj')})$; $j, j' = 1, 2, \dots, k$ is proved below to be k^2 - variate normal, following the theorem of Hoeffding [9, p.310] on U-statistics for non-identically distributed vectors, which can be extended as follows.

Theorem E. Let $Z_1, Z_2, \dots, Z_n$ be n independent random vectors of k components, Z_i having the c.d.f. $F_i(Z) = F_i(Z^{(1)}, Z^{(2)}, \dots, Z^{(k)})$. Let

$\phi^j(Z_1, Z_2, \dots, Z_h)$, $j = 1, 2, \dots, g$ be functions symmetric in its h vector arguments which do not involve n , and let

$$\begin{aligned} \psi_{1(i_v) i_1 i_2 \dots i_{v-1} i_{v+1} \dots i_h}^j(Z_{i_v}) &= E \{ \phi^j(Z_{i_1}, \dots, Z_{i_h}) \mid Z_{i_v} \} \\ &= E \{ \phi^j(Z_{i_1}, \dots, Z_{i_h}) \} \end{aligned}$$

$$\bar{\Psi}_{1(v)}^{(j)}(Z_v) = \left(\frac{n-1}{h-1} \right)^{-1} \sum_{(i_1, i_2, \dots, i_{h-1})} \psi_{1(v) i_1 i_2 \dots i_{h-1}}^j(Z_v) \quad (v = 1, 2, \dots, n)$$

where $(i_1, i_2, \dots, i_{h-1})$ extends over $1 \leq i_1, i_2, \dots, i_{h-1} \leq n$, with no two i 's equal and no i equal to v . Suppose there exists a number A such that for every $n=1, 2, \dots$ and $j=1, 2, \dots, g$,

$$\int \dots \int \{ \phi^j(Z_1, Z_2, \dots, Z_h) \}^2 dF_{i_1}(Z_1) \dots dF_{i_h}(Z_h) < A$$

$$(1 \leq i_1, i_2, \dots, i_h \leq n, \text{ no two } i\text{'s equal})$$

$$E \mid \bar{\Psi}_{1(v)}^j(Z_v) \mid^3 < \infty \quad (v = 1, 2, \dots, n; j = 1, 2, \dots, g) \text{ and}$$

$$\lim_{n \rightarrow \infty} \sum_{v=1}^n E \mid \sum_{j=1}^g c_j \bar{\Psi}_{1(v)}^j(Z_v) \mid^3 / \left[\sum_{v=1}^n E \{ \sum_{j=1}^g c_j \bar{\Psi}_{1(v)}^j(Z_v) \}^2 \right]^{3/2} = 0$$

for arbitrary constants $(c_1, c_2, \dots, c_g)$. Then, as $n \rightarrow \infty$, the joint distribution of $\{(U^j - EU^j)/\sqrt{\text{Var } U^j}\}$, $j = 1, 2, \dots, g$ tends to a g -variate normal distribution with null mean vector.

If $h = k$, and $g = k^2$, it is easily verified in our special case that,

$$\begin{aligned} \int \dots \int \{ \phi^{(jj)}(Z_1, Z_2, \dots, Z_k) \}^2 dF_{i_1}(Z_1) dF_{i_2}(Z_2) \dots dF_{i_k}(Z_k) \\ \leq (k-1)^2 \end{aligned}$$

$$E \left| \bar{\Psi}_{1(v)}^{(jj')} (Z_v) \right|^3 \leq 8(k-1)^3 < \infty$$

It follows from above that,

$$(3.2.8) \quad \sum_{v=1}^n E \left| \sum_{(j,j')} c_{jj'} \bar{\Psi}_{1(v)}^{(jj')} (Z_v) \right|^3 < M_1 n$$

where $0 < M_1 < \infty$ is independent of n . Consider $v \in S_{jj'}$. If $Z_{i_1}, Z_{i_2}, \dots, Z_{i_{k-1}}$ are $(k-1)$ arbitrary vectors different from Z_v , then

$$\begin{aligned} \Psi_{1(v)i_1 i_2 \dots i_{k-1}}^{(jj')} (Z_v) &= E \left\{ \phi^{(jj')} (Z_v, Z_{i_1}, \dots, Z_{i_{k-1}}) \mid Z_v \right\} \\ &= E \left\{ \phi^{(jj')} (Z_v, Z_{i_1}, \dots, Z_{i_{k-1}}) \right\} \\ &= \sum_{j'=1}^k \left[\sum_{i_{j'} \neq v} \left\{ E(\delta(Z_{vj'}, Z_{i_{j'}}, j')) \mid Z_v \right\} - E\delta(Z_{vj'}, Z_{i_{j'}}, j') \right\} \\ &\quad + \sum_{i_{j'} \neq v \in S_{jj'}} \left\{ E(\delta(Z_{i_{j'}j}, Z_{vj'}, j)) \mid Z_v \right\} - E\delta(Z_{i_{j'}j}, Z_{vj'}, j') \right\} \Big] / k! \end{aligned}$$

Thus,

$$\begin{aligned} \bar{\Psi}_{1(v)}^{(jj')} (Z_v) &= \sum_{j'=1}^k \left[\sum_{i \neq v} \left\{ E(\delta(Z_{vj'}, Z_{ij'}, j')) \mid Z_v \right\} - E\delta(Z_{vj'}, Z_{ij'}, j') \right\} \\ &\quad + \sum_{i \neq v \in S_{jj'}} \left\{ E(\delta(Z_{ij'}, Z_{vj'}, j)) \mid Z_v \right\} - E\delta(Z_{ij'}, Z_{vj'}, j') \right\} \Big] / k! \\ &= \left[rk \left\{ G(Z_{ij'}) - 1/2 \right\} + (r \text{ or } \rho) \sum_{j'=1}^k \left\{ 1/2 - G(Z_{vj'}) \right\} + O(\sqrt{n}) \right] / k! \\ &= n \left[m \left\{ G(Z_{ij'}) - 1/2 \right\} + (m/k \text{ or } m(m-1)/k(k-1)) \sum_{j'=1}^k \left\{ 1/2 - G(Z_{vj'}) \right\} + O(n^{-1/2}) \right] / k! \end{aligned}$$

where the first or the second alternative is chosen according as $j = j'$, or

$j \neq j'$. It is seen from above that the variance of any linear combination of the left-hand side is bounded away from zero. Thus, we can find $M_2 > 0$, such that,

$$\sum_{v=1}^n E \left\{ \sum_{(j,j')} c_{jj', \bar{\psi}_{1(v)}^{(jj')}} (Z_v) \right\}^2 \geq M_2 n$$

which with (3.2.8) verifies the last condition of theorem E. Thus all conditions of the theorem are satisfied, proving that $\{ U^{(jj')} - EU^{(jj')} \} / \sqrt{\text{Var } U^{(jj')}} ; j, j' = 1, 2, \dots, k$ have a joint k^2 - variate normal distribution, as $n \rightarrow \infty$. The asymptotic joint normality of $\{ U^{(jj')} - EU^{(jj')} \} / \sqrt{\text{Var } U^{(jj')}} ; j, j' = 1, 2, \dots, k$ follows from the results on p.21, using the fact that $n \cdot \text{Var } U^{(jj')}$ exists - which is verified below. The mean vector and variance-covariance matrix of the limiting normal distribution, following the results of Hoeffding, can now be calculated by computing their values for a fixed n ; and then proceeding to the limits as n tends to infinity. The existence of the mean vector and the variance-covariance matrix and their limiting values have been shown below.

Using the relation $r_{ij} = \sum_{u=1}^n \sum_{v=1}^k \delta(Z_{ij}, Z_{uv}) + 1$, it is easily verified that,

$$E(r_{ij}) = nm/2 - \sqrt{n} \cdot m \left[\int_{-\infty}^{\infty} \{ G'(u) \}^2 du \right] (\theta_j - \bar{\theta}_i) + o(1)$$

Thus,

$$\sum_{i \in S_{jj'}} E(r_{ij}) = nmr/2 - \sqrt{n} \rho k \left[\int_{-\infty}^{\infty} \{ G'(u) \}^2 du \right] (\theta_j - \bar{\theta}) + o(n)$$

$$\sum_{i \in S_{jj'}} \sum_{j' \neq j=1}^k E(r_{ij'}) = nmr(k-1)/2 + \sqrt{n} \rho k \left[\int_{-\infty}^{\infty} \{ G'(u) \}^2 du \right] (\theta_j - \bar{\theta}) + o(n)$$

Using (3.2.4) and (3.2.5), this gives

$$(3.2.9) \quad \lim_{n \rightarrow \infty} E(T_{nj}) = \mu_j = -\sqrt{3(m-1)} \left[\int_{-\infty}^{\infty} \{G'(u)\}^2 du \right] (\theta_j - \bar{\theta}) / \sqrt{(k-1)(1-3\lambda)}$$

Similarly,

$$\begin{aligned} \text{Var}(r_{ij}) &= E \left\{ \sum_{u=1}^n \sum_{v=1}^k \delta(Z_{ij}, Z_{uv}) \right\}^2 - E^2 \left\{ \sum_{u=1}^n \sum_{v=1}^k \delta(Z_{ij}, Z_{uv}) \right\} \\ &= \sum_{i'} \sum_u \sum_s \sum_v \left[E \{ \delta(Z_{ij}, Z_{i's}) \delta(Z_{ij}, Z_{uv}) \} \right. \\ &\quad \left. - E \{ \delta(Z_{ij}, Z_{i's}) \} E \{ \delta(Z_{ij}, Z_{uv}) \} \right] \\ &= n^2 m^2 / 12 + O(n^{3/2}) \end{aligned}$$

$$\text{Cov}(r_{ij}, r_{ij'}) = n^2 m^2 (\lambda - 1/4) + O(n^{3/2})$$

Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} (\text{Var } T_{nj}) &= \lim_{n \rightarrow \infty} \left\{ 3(k-1) \right\} / \left\{ n^3 m^4 (m-1)(1-3\lambda) \right\} \cdot \text{Var} \left\{ (m-1) \sum_{i \in S_j} r_{ij} \right. \\ &\quad \left. - \sum_{j' \neq j=1}^k \sum_{i \in S_{jj'}} r_{ij'} \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ 3(k-1) / n^3 m^4 (m-1)(1-3\lambda) \right\} \cdot \{ m(m-1)r(m^2 n^2 / 12) - m(m-1)r \{ m^2 n^2 (\lambda - 1/4) \} \} \\ (3.2.10) \quad &= (1-1/k) \end{aligned}$$

Since the expansion of $\text{Var} \left\{ (m-1) \sum_{i \in S_j} r_{ij} - \sum_{j' \neq j=1}^k \sum_{i \in S_{jj'}} r_{ij'} \right\}$

using the formula $\text{Var} \left(\sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{i>i'} \text{Cov}(X_i, X_{i'})$ gives

$m(m-1)r$ terms of each of the type $\text{Var}(r_{ij})$ and $\text{Cov}(r_{ij}, r_{ij'})$. The only other terms of the type $\text{Cov}(r_{ij}, r_{i'j'})$ when $i \neq i'$, cancel out. The

limiting covariance term is similarly calculated or follows from the equation $\sum_{j=1}^k T_{nj} = 0$ coupled with the fact that the above limit is independent of $(\theta_1, \theta_2, \dots, \theta_k)$, so that,

$$(3.2.11) \quad \lim_{n \rightarrow \infty} \text{Cov} (T_{nj}, T_{nj'}) = - \lim_{n \rightarrow \infty} \text{Var} (T_{nj}) / (k-1) = -1/k$$

A proof of asymptotic normality together with results (3.2.9), (3.2.10), and (3.2.11) completes the proof of Lemma 3.1.

Lemma 3.2. Under the assumptions of Theorem 3.1, the conditional variance-covariance matrix of $T_n = (T_{n1}, T_{n2}, \dots, T_{nk})$, given by (3.2.2) tends in probability, as $n \rightarrow \infty$, to the unconditional variance-covariance matrix given by Lemma 3.1.

Proof. Since the expectation of the conditional variance-covariance matrix is the unconditional variance-covariance matrix; the conclusion of the lemma will follow from Chebyshev's inequality if we prove that the variance of each element of the conditional variance-covariance matrix tends to zero, under the alternatives, as $n \rightarrow \infty$. This is proved as follows.

$$\begin{aligned} \phi_{jj} &= \widetilde{\text{Var}} \left\{ \sum_{i \in S_j} r_{ij} / n^{3/2} \right\} = \sum_{i \in S_j} \left\{ \sum_{j=1}^k r_{ij}^2 / m - \left(\sum_{j=1}^k r_{ij} / m \right)^2 \right\} / n^3 \\ \text{Var } \phi_{jj} &= \text{Var} \left[\sum_{i \in S_j} \left\{ \sum_{j=1}^k r_{ij}^2 / m - \left(\sum_{j=1}^k r_{ij} / m \right)^2 \right\} \right] / n^6 \\ &\leq 2 \left[\text{Var} \left\{ \sum_{i \in S_j} \sum_{j=1}^k r_{ij}^2 \right\} / m^2 + \text{Var} \left\{ \sum_{i \in S_j} \left(\sum_{j=1}^k r_{ij} \right)^2 \right\} / m^4 \right] / n^6 \\ &\rightarrow 0, \text{ as } n \rightarrow \infty, \end{aligned}$$

since each of the two terms above tend to zero by virtue of the properties

$$(i) \text{Cov} (r_{11}^2, r_{21}^2) = O(n^3), \quad (ii) \text{Cov}(r_{11}^2, r_{21}r_{22}) = O(n^3),$$

$$(iii) \text{Cov} (r_{11}r_{12}, r_{21}r_{22}) = O(n^3)$$

Proof of Theorem 3.1. Let $T_n^* = (T_{n1}, T_{n2}, \dots, T_{n(k-1)})$ and $\Lambda_n^* = || \delta_{jj'} - 1/k ||$; $j, j' = 1, 2, \dots, (k-1)$ denote their associated variance-covariance matrix. Then by virtue of Lemma 3.2, using the following theorem given in Cramér; both W_n and $T_n^* \Lambda_n^{*-1} T_n^{*'}$ have the same asymptotic distribution. Following theorem C [17] the latter has a non-central chi-square distribution with $(k-1)$ - degrees of freedom and non-centrality parameter $\Delta_w = \mu_n^* \Lambda_n^{*-1} \mu_n^{*'} = \sum_{j=1}^k \mu_j^2$, where $\mu_n^* = (\mu_1, \mu_2, \dots, \mu_{k-1})$ - which agrees with the value given by (3.2.3). This proves the theorem.

Theorem F. Let $\zeta_1, \zeta_2, \dots$ be a sequence of random variables, with the distribution functions $F_1, F_2, \dots$. Suppose that $F_n(x)$ tends to $F(x)$, as $n \rightarrow \infty$. Let $\eta_1, \eta_2, \dots$ be another sequence of random variables, and suppose that η_n converges in probability to a constant c . Putting $Z_n = (\zeta_n / \eta_n)$, if $c > 0$, the distribution function of Z_n tends to $F(cx)$.

3.3. Partially Balanced Incomplete Block Designs.

The results of this section, contained in Theorem 3.1(A) below, are direct extensions of the results of Theorem 3.1. Let our experimental layout be a partially balanced incomplete block design with parameters $(n, k, m, r, \rho_{jj'})$ where

$\rho_{jj'}$ = number of blocks in which both the j^{th} and j'^{th} treatments occur together,

and the other symbols have the same meaning as in section 3.2.

We further assume that,

$$(3.3.1) \quad \lim_{n \rightarrow \infty} \{ \rho_{jj'} / n \} = p_{jj'} > 0$$

Then the equations (3.2.1) can be modified as,

$$(3.3.2) \quad n.m = r.k, \quad r(m-1) = \sum_{j' \neq j=1}^k \rho_{jj'}, \quad n \geq k \geq m, \\ r \leq n$$

We now state,

Theorem 3.1(A). Assume for each n, the truth of the hypothesis
 K_n , and that the conditions of Theorem 3.1 and the condition (3.3.1)
are satisfied. Then, for the above partially balanced incomplete block
design, the statistic W_n defined by (1.4.2), converges in distribution,
as $n \rightarrow \infty$, to $\chi^2_{k-1}(\Delta'_W)$, where,

$$(3.3.3) \quad \Delta'_W = \left\{ 3(m-1)(k-1)/k^2(1-3\lambda) \right\} \left[\int_{-\infty}^{\infty} \{ G'(u) \}^2 du \right]^2 \\ \times \sum_{j=1}^k \left\{ \theta_j - k/m(m-1) \sum_{j' \neq j=1}^k p_{jj'} \theta_{j'} \right\}^2$$

Proof Let

$$(3.3.4) \quad T_{nj} = \left\{ \sqrt{3(k-1)}/m\sqrt{(m-1)(1-3\lambda)} \right\} \left[R_{j-1/m} \sum_{j'=1}^k \sum_{i \in S_{jj'}} r_{ij'} \right] / n^{3/2}$$

and

$$(3.3.5) \quad \mu_j = - \left\{ \sqrt{3(m-1)(k-1)}/k\sqrt{1-3\lambda} \right\} \left[\int_{-\infty}^{\infty} \{ G'(u) \}^2 du \right] \left\{ \theta_j - k/m(m-1) \right. \\ \left. \cdot \sum_{j' \neq j=1}^k p_{jj'} \theta_{j'} \right\}$$

where $S_{jj'}$ is the same as in Lemma 2.1. Following the lines of proof of Lemma 3.1, we can prove that the random vector $T_n = (T_{n1}, T_{n2}, \dots, T_{nk})$ has an asymptotic k - variate normal distribution with mean vector

$\underline{\mu} = (\mu_1, \mu_2, \dots, \mu_k)$ and variance - covariance matrix $\underline{\Lambda}$. It can be proved as in Lemma 3.2 that the conditional variance - covariance matrix of T_n , given by (3.2.2) tends in probability, as $n \rightarrow \infty$, to the unconditional variance-covariance matrix $\underline{\Lambda}$. The proof of the theorem thus can be accomplished exactly in the same manner as Theorem 3.1 and need not be repeated here.

3.4 "Balanced" Incomplete Blocks With p Observations Per Cell

The results of this section are an extension of the corresponding results of section 3.2. Suppose, in the experimental design of section 3.2, p observations are available for every treatment that is present in a block i.e. $m_{ij} = 0$ or p. The notations of this section have the same meaning as the corresponding notations of section 3.2. Further, let D_{ij} denote the sum of the ranks under the j^{th} treatment in the i^{th} block. Define

$$(3.4.1) \quad T_{nj} = \left[\sqrt{3(kp-1)} / \{m(np)^{3/2} \sqrt{(mp-1)(1-3\lambda)}\} \right] \left[R_j - \sum_{j'=1}^k \sum_{i \in S_{jj'}} D_{ij} / m \right]$$

and

$$(3.4.2) \quad \mu_j = - \{ (m-1) \sqrt{3p(kp-1)} \} / \{ (k-1) \sqrt{(mp-1)(1-3\lambda)} \} \\ \times \left[\int_{-\infty}^{\infty} \{ G'(u) \}^2 du \right] (\theta_j - \bar{\theta})$$

for $j = 1, 2, \dots, k$. It is easily verified that $\left(\sum_{j=1}^k T_{nj}^2 \right)$ is asymptotically equivalent to the statistic W_n of (1.4.2).

Theorem 3.1 (B). Assume for each n the truth of the hypothesis K_n ,
and that the conditions of Theorem 3.1 are satisfied. Then for the
above experimental design, the statistic W_n defined by (1.4.2), converges
in distribution, as $n \rightarrow \infty$, to $\chi_{k-1}^2(\Delta_W'')$, where

$$(3.4.3) \quad \Delta_W'' = \sum_{j=1}^k \mu_j^2 = \{ 3p(kp-1)(m-1)^2 \} / \{ (mp-1)(1-3\lambda)(k-1)^2 \} \\ \times \left[\int_{-\infty}^{\infty} \{ G'(u) \}^2 du \right]^2 \sum_{j=1}^k (\theta_j - \bar{\theta})^2$$

Proof. The asymptotic normality of $\underline{T}_n = (T_{n1}, T_{n2}, \dots, T_{nk})$, its limiting mean vector and variance - covariance matrix are deduced from the results of section 3.3.

This is done by associating separate θ_j 's with the p observations in a cell and noting that our experimental design reduces to a partially balanced incomplete block with two associate classes, having parameters $(n, kp, mp, r, \rho_{jj'})$ where

n = number of blocks
 kp = number of treatments
 mp = block size
 r = number of replications of each treatment
 $\rho_{jj'}$ = number of blocks in which both the j^{th} and j'^{th} treatments occur together.

This is r or ρ according as j and j' do or do not occur in the same cell. The joint distribution of $\underline{T}_n = (T_{n1}, T_{n2}, \dots, T_{nk})$ follows by a linear transformation and the results of Theorem 3.1(B) follows as before after setting the θ_j 's within the same cell to be equal.

Remark It may be observed that in Theorems 3.1, 3.1(A) and 3.1(B), the asymptotic distribution of the statistic W_n is the same as the asymptotic distribution of $\left(\sum_{j=1}^k T_{nj}^2 \right)$. The later expression with λ replaced by a consistent estimator may therefore be proposed as an alternative test statistic. Many such consistent estimators of λ are provided by the results

of Lemma 3.2 (or its analogues in Theorem 3.1(A) or 3.1(B)). This statistic may more easily be applied. However, whereas the convergence of W_n to the chi-square distribution is uniform in configuration; the same property may not hold in case of the later statistic.

CHAPTER IV

COMPARISON OF EFFICIENCIES

4.1 The Concept of Asymptotic Efficiency

The concept of asymptotic (local) efficiency, originally given by Pitman and subsequently studied by Noether [15], seems a satisfactory criterion for comparing two consistent test procedures. Suppose S and S^* be two consistent test procedures and $\{k_n\}$ be a sequence of alternatives approaching the null hypothesis. For a fixed n and a pre-assigned level of significance α , let $N_1(n)$ and $N_2(n)$ respectively denote the sample sizes, required to give the same power β to both the tests S and S^* . The asymptotic efficiency of S relative to S^* , for the sequence of alternatives $\{k_n\}$, is defined to be

$$e_{S,S^*} = e_{S,S^*}(\alpha, \beta, \{k_n\}) = \lim_{n \rightarrow \infty} N_2(n) / N_1(n)$$

if the limit of the right-hand side exists. In practice, the limit is often independent of the choice of (α, β) ; but the sequence of alternatives have to be suitably chosen to find its value.

4.2 Evaluation of Asymptotic Efficiency

In this section we give exact expressions for the asymptotic (Pitman) efficiencies of our tests relative to the competing test procedures, namely, the classical F -test and the Durbin's (or appropriate separate ranking) test procedure. There will be some overlap of the results stated here with those in section 4 of Mehra and Sarangi [14]. It has been used by Andrews [1] and pointed out by Hannan [6] that if under the alternatives both the test statistics have non-central chi-square distributions with the same degrees of freedom, then the

asymptotic relative efficiency of the corresponding test procedures is given by the ratio of the corresponding non-centrality parameters. We discuss each of the previous cases separately.

I. Balanced Incomplete Block Design

The classical F -statistic is distributed in the limit as $n \rightarrow \infty$, as $\chi^2_{k-1}(\Delta_F)$ where,

$$(4.2.1) \quad \Delta_F = (m-1) \left\{ \sum_{j=1}^k (\theta_j - \bar{\theta})^2 \right\} / (k-1)\sigma^2$$

if the variance σ^2 for the parent distribution exists. The corresponding Durbin's test statistic W_n^* , based on independent ranking within each block is given by

$$W_n^* = \{1^2/k\rho(m+1)\} \sum_{j=1}^k \{R_j^* - \frac{r}{2}(m+1)\}^2$$

Its large sample behaviour, as $n \rightarrow \infty$, is given by the following theorem.

Theorem G. Assuming, for each fixed n , the truth of the hypothesis K_n , if (i) $F(x)$ possesses a continuous derivative $F'(x)$ a.e. and (ii) there exists a function $b(x)$ such that $[F(x+\theta) - F(x)] / \theta \leq b(x)$ where

$$\int_{-\infty}^{\infty} b(x) dF(x) < \infty, \text{ then the statistic } W_n^* \text{ converges, as } n \rightarrow \infty, \text{ to } \chi^2_{k-1}(\Delta_{w*}),$$

where

$$(4.2.2) \quad \Delta_{w*} = \{12m(m-1)/(m+1)(k-1)\} \left\{ \int_{-\infty}^{\infty} [F'(x)]^2 dx \right\} \sum_{j=1}^k (\theta_j - \bar{\theta})^2$$

The results of (4.2.1) and the preceding theorem are contained in van Elteren and Noether [19].

From (3.2.3), (4.2.1) and (4.2.2), taking the ratio of the non-centrality parameters, we get

$$(4.2.3) \quad e_{w, w^*} = \left[\frac{(m+1)}{4m(1-3\lambda)} \right] \left[\frac{\int_{-\infty}^{\infty} \{G'(x)\}^2 dx}{\int_{-\infty}^{\infty} \{F'(x)\}^2 dx} \right]^2$$

$$(4.2.4) \quad e_{w, F} = \left[\frac{3\sigma^2}{(1-3\lambda)} \right] \left[\int_{-\infty}^{\infty} \{G'(x)\}^2 dx \right]^2$$

where λ is defined by (3.2.3).

Explicit evaluation of the expressions (4.2.3) and (4.2.4) seems difficult in most cases; except for the normal form of the parent distribution, for which we give computations below. For the special case where $m=2$, it is easily checked that $\lambda = 1/6$ and the above expressions can be evaluated. Some values of $e_{w, F}$, assuming different forms of F are given below.

	Normal	Uniform	Exponential	Double Exponential
F	.955	.889	1.5	1.172

The values of $e_{w, F}$ from the above table indicate the rather high efficiency of our test relative to the F-test.

Efficiency for Normal Parent Distribution

Suppose $F'(x) = (\sqrt{2\pi}\sigma)^{-1} \exp \left\{ -\frac{1}{2} \left(\frac{x-a}{\sigma} \right)^2 \right\}$

Then, by direct computation, we get

$$G'(x) = \{\sqrt{m}/\sigma \sqrt{2\pi(m-1)}\} \exp \{-mx^2/2(m-1)\sigma^2\}$$

$$\int_{-\infty}^{\infty} \{F'(x)\}^2 dx = \frac{1}{2\sigma\sqrt{\pi}} \quad \text{and}$$

$$\int_{-\infty}^{\infty} \{G'(x)\}^2 dx = \sqrt{m} / 2\sigma\sqrt{\pi(m-1)}$$

If x_{ij} ($i=1,2,3$; $j=1,2,\dots,m$) be $3m$ independent random variables distributed identically with c.d.f. $F(x)$ and $\bar{x}_i = \frac{1}{m} \sum_{j=1}^m x_{ij}$, then

$$\begin{aligned} \lambda &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(u)G(v)dH(u,v) = \Pr[(x_{11} - \bar{x}_1) - (x_{21} - \bar{x}_2) > 0, \\ &\quad (x_{12} - \bar{x}_1) - (x_{31} - \bar{x}_3) > 0] \\ &= \Pr [u > 0, v > 0] \end{aligned}$$

where (u,v) have a joint distribution that is bivariate normal with zero means, common variance $\{2\sigma^2(m-1)/m\}$ and covariance $\{-\sigma^2/m\}$. Thus,

$$\begin{aligned} \lambda &= \int_0^{\infty} \int_0^{\infty} \frac{m}{2\pi\sigma^2\sqrt{4(m-1)^2-1}} \exp\left\{-\frac{m(m-1)}{\sigma^2[4(m-1)^2-1]} \left(x^2 + y^2 + \frac{xy}{(m-1)}\right)\right\} dx dy \\ &= \frac{m}{2\pi\sqrt{4(m-1)^2-1}} \int_0^{\pi/2} \left[\int_0^{\infty} \exp\left\{-\frac{m(m-1)r^2}{4(m-1)^2-1} \left(1 + \frac{\sin\theta \cos\theta}{(m-1)}\right)\right\} r dr \right] d\theta \\ &= \frac{\sqrt{4(m-1)^2-1}}{4\pi} \int_0^{\pi/2} \frac{d\theta}{(m-1) + \sin\theta \cos\theta} \\ &= \frac{\sqrt{4(m-1)^2-1}}{2\pi} \int_0^{\pi/2} \frac{d\beta}{2(m-1) + \sin\beta} \\ &= \frac{1}{\pi} \arctan \sqrt{(2m-3)/(2m-1)} \end{aligned}$$

Substituting these in (4.2.3) and (4.2.4), we get

$$(4.2.5) \quad \begin{cases} e_{w,w^*}(\text{normal}) = \left[\frac{(m+1)}{4(m-1)} \right] \left[1 - \left(\frac{3}{\pi} \right) \arctan \sqrt{(2m-3)/(2m-1)} \right]^{-1} \\ e_{w,F}(\text{normal}) = \left[\frac{3m}{4\pi(m-1)} \right] \left[1 - \left(\frac{3}{\pi} \right) \arctan \sqrt{(2m-3)/(2m-1)} \right]^{-1} \end{cases}$$

It is easily checked that (4.2.5) can be rewritten as

$$(4.2.6) \quad \begin{cases} e_{w,w^*}(\text{normal}) = \left[\frac{(5-3\alpha^2)}{4(1+\alpha^2)} \right] \left[1 - \left(\frac{3}{\pi} \right) \arctan \alpha \right]^{-1} \\ e_{w,F}(\text{normal}) = \left[\frac{3(3-\alpha^2)}{4\pi(1+\alpha^2)} \right] \left[1 - \left(\frac{3}{\pi} \right) \arctan \alpha \right]^{-1} \end{cases}$$

where $\alpha^2 = (2m-3)/(2m-1)$. Differentiating by the product rule,

$$\begin{aligned} \frac{d}{d\alpha} \{e_{w,w^*}(\text{normal})\} &= \left[12\alpha\pi / \{(1+\alpha^2)(\pi-3 \arctan \alpha)\}^2 \right] \left[(5-3\alpha^2)/16 + \right. \\ &\quad \left. + \arctan \alpha - \frac{\pi}{3} \right] < 0 \end{aligned}$$

since the quantity within the square brackets has a negative derivative with respect to α and thus has a maximum when α has the minimal value of $1/\sqrt{3}$ (for $m=2$) - which can be verified to be negative. Thus $e_{w,w^*}(\text{normal})$ given by (4.2.5), attains a maximum of $3/2$ for $m=2$, and monotonically decreases to 1, as m tends to infinity. Similarly,

$$\begin{aligned} \frac{d}{d\alpha} \{e_{w,F}(\text{normal})\} &= \left[9 / \{ 4\alpha(1+\alpha^2)^2(\pi-3 \arctan \alpha)^2 \} \right] \left[(3-\alpha^2)/\alpha - 8\pi/3 + \right. \\ &\quad \left. + 8 \arctan \alpha \right] \end{aligned}$$

The quantity within the square brackets can be verified to be positive for $m=2$ when $\alpha = 1/\sqrt{3}$, negative for $m=3$ when $\alpha = \sqrt{3/5}$ and its derivative with respect to α can be checked to be negative. Thus, considering only integral values of m , $e_{w,F}(\text{normal})$ given by (4.2.5) attains a minimum of $3/\pi$ for $m=2$, a maximum for $m=3$ and monotonically decreases to $3/\pi$ as m tends to infinity. Thus,

$$(4.2.7) \quad e_{w,F} \geq 3/\pi \text{ for all } m.$$

The following table gives some values of the efficiency expressions (4.2.5)

m	2	3	4	5	∞
$e_{w,w^*}(\text{normal})$	1.5	1.355	1.263	1.210	1
$e_{w,F}(\text{normal})$	$3/\pi = .9549$	.9662	.9646	.9631	$3/\pi$

The behaviour of $e_{w,w^*}(\text{normal})$ verifies the intuitively clear fact that the gain in efficiency due to inter-block comparisons, by using the joint ranking procedure instead of the separate ranking one, decreases to zero as the size of the blocks tends to grow larger. A statement in these lines was made by Hodges-Lehmann ([8], Pp 495).

Efficiency for large m

The preceding remark that e_{w,w^*} tends to 1, as $m \rightarrow \infty$, if the parent distribution is normal holds, in fact for all cases where the parent distribution has a finite expectation. This result was proved in [14] and we give a proof here for convenience.

Theorem 4.1 If (i) for the parent distribution $F(x)$, $\int_{-\infty}^{\infty} x dF(x) = a < \infty$
and (ii) there exists a m_0 such that for all $m > m_0$, there exists a
function $b(u)$ such that $[\{G^m(u+\theta) - G^m(u)\}/\theta] = [\{G(u+\theta) - G(u)\}/\theta] \leq b(u)$,
where $\int_{-\infty}^{\infty} b^2(u) du < \infty$, then $e_{w,w^*}(F)$ given by (4.2.3) satisfies

$$\lim_{n \rightarrow \infty} e_{w,w^*}(F) = 1.$$

Proof: Suppose x_{ij} ($i = 1, 2, \dots; j = 1, 2, \dots, m$) be independent r.v.'s with identical c.d.f. $F(x)$ and let $\bar{x}_i = \frac{1}{m} \sum_{j=1}^m x_{ij}$. Since $F(x)$ has a finite expectation, $\bar{x}_i$ tends to a almost surely for all i by the strong law of large numbers and so

$$\lim_{m \rightarrow \infty} \lambda = \lim_{m \rightarrow \infty} [(x_{11} - \bar{x}_1) > (x_{21} - \bar{x}_2), (x_{12} - \bar{x}_1) > (x_{31} - \bar{x}_3)]$$

$$= \Pr[x_{11} > x_{21}, x_{12} > x_{31}] = 1/4.$$

Further,

$$\lim_{m \rightarrow \infty} G(x) = \lim_{m \rightarrow \infty} \Pr[x_{11} - \bar{x}_1 \leq x] = F(x + a)$$

which implies, using condition (ii) and the dominated convergence theorem,

$$\lim_{m \rightarrow \infty} \left[\int_{-\infty}^{\infty} \{G'(x)\}^2 dx / \int_{-\infty}^{\infty} \{F'(x)\}^2 dx \right] = 1.$$

This proves the theorem.

Remark: The following example shows that the condition of finiteness of the expectation is essential for the conclusion of Theorem 4.1.

Suppose, the r.v. has a Cauchy distribution of the form

$$F'(a, b, x) = \frac{b}{\pi} \cdot \frac{1}{b^2 + (x-a)^2}, \quad b > 0$$

so that the expectation does not exist. Since it is easily seen that

e_{w, w^*} as given by (4.2.3) is independent of any change of scale or

location of the r.v., we assume without any loss of generality that $a=0$

and $b=1$. In this case, $(x_{11} - \bar{x}_1)$ is a Cauchy variable having density

function $F'(0, 2(m-1)/m, x)$ which tends to $F'(0, 2, x)$ as $m \rightarrow \infty$. So

$$\lim_{m \rightarrow \infty} G'(x) = F'(0, 2, x).$$

It is easily verified that,

$$\lim_{m \rightarrow \infty} \left[\int_{-\infty}^{\infty} \{G'(x)\}^2 dx / \int_{-\infty}^{\infty} \{F'(x)\}^2 dx \right] = 1/2$$

$$\lim_{m \rightarrow \infty} \lambda = \lim_{m \rightarrow \infty} \Pr [(x_{11} - \bar{x}_1) > (x_{21} - \bar{x}_2), (x_{12} - \bar{x}_1) > (x_{31} - \bar{x}_3)]$$

$$= \lim_{m \rightarrow \infty} \Pr \left[\left(\sum_{j=3}^m x_{1j} \right) / m > \left(1 - \frac{1}{m} \right) x_{21} - \left(\sum_{j=2}^m x_{2j} \right) / m - \left(1 - \frac{1}{m} \right) x_{11} + \frac{x_{12}}{m}, \right.$$

$$\left. \left(\sum_{j=3}^m x_{1j} \right) / m > \left(1 - \frac{1}{m} \right) x_{31} - \left(\sum_{j=2}^m x_{3j} \right) / m - \left(1 - \frac{1}{m} \right) x_{12} + \frac{x_{11}}{m} \right]$$

$$= \Pr [x > u, x > v]$$

where x, u, v are independent Cauchy r.v.'s of the type $F'(0,1,x), F'(0,3,x)$ and $F'(0,3,x)$ respectively. The right-hand side can be simplified to

$$1/4 + 1/\pi^3 \int_{-\infty}^{\infty} \{(\arctan \frac{x}{3})^2/(1+x^2)\} dx.$$

Substituting in (4.2.3), we get

$$(4.2.8) \quad \lim_{m \rightarrow \infty} e_{w, w^*}(\text{Cauchy}) = [4 - \frac{48}{\pi^3} \int_{-\infty}^{\infty} \{(\arctan \frac{x}{3})^2/(1+x^2)\} dx]^{-1}$$

For evaluating the integral, consider

$$I(\beta) = \int_{-\infty}^{\infty} \{(\arctan \frac{x}{\beta})^2/(1+x^2)\} dx, \quad \beta > 1.$$

$$\text{Let} \quad I(\beta) = \frac{dI(\beta)}{d\beta} = 2 \int_{-\infty}^{\infty} \{x \arctan (-x/\beta)\} / \{(1+x^2)(\beta^2+x^2)\} dx$$

To find $I(\beta)$, consider

$$I_z(\beta) = \int_{\Gamma} \{z \log (\beta-iz)\} / \{(1+z^2)(\beta^2+z^2)\} dz$$

where Γ refers to the contour bounded by the x -axis and the upper half of the circle with centre at the origin and radius R . Calculating the residues at $z = i, \beta i$; since the integral vanishes along the circular arc as $R \rightarrow \infty$, we get

$$\int_{-\infty}^{\infty} \frac{x \log(\beta-ix)}{(1+x^2)(\beta^2+x^2)} dx = \frac{i\pi}{\beta^2-1} \log \frac{\beta+1}{2\beta}$$

Taking the imaginary parts of either side, we get

$$\int_{-\infty}^{\infty} \{x \arctan (-x/\beta)\} / \{(1+x^2)(\beta^2+x^2)\} dx = \frac{\pi}{\beta^2-1} \log \frac{\beta+1}{2\beta}$$

$$\begin{aligned}
 \text{Thus, } I(\beta) &= 2\pi \int \frac{1}{\beta^2-1} \log \frac{\beta+1}{2\beta} d\beta \\
 &= 2\pi \int \frac{1}{\beta^2} \left(1 - \frac{1}{\beta^2}\right)^{-1} \log\left(1 + \frac{1}{\beta}\right) d\beta - \pi \log 2 \log \frac{\beta-1}{\beta+1} \\
 &= 2\pi \left(-\frac{1}{2\beta^2} + \frac{1}{6\beta^3} - \frac{1}{3\beta^4} + \frac{3}{20\beta^5} - \frac{23}{90\beta^6} \dots\right) - \pi \log 2 \log \frac{\beta-1}{\beta+1}
 \end{aligned}$$

by using the power series expansions for both $\left(1 - \frac{1}{\beta^2}\right)^{-1}$ and $\log(1 + 1/\beta)$ and the fact that the right-hand side vanishes as $\beta \rightarrow \infty$. Specially if $\beta=3$, by direct computation

$$I(3) = 1.1749$$

Substituting in (4.2.8),

$$\lim_{m \rightarrow \infty} e_{w, w^*}(\text{Cauchy}) = .458.$$

It is easily seen that for $m=2$, $e_{w, w^*}(\text{Cauchy}) = .75$. Thus the efficiency of the joint ranking procedure relative to the separate ranking one decreases from .75 to .458 as m increases from 2 to infinity - presumably staying below unity in the entire range.

Remark Thus joint ranking procedure, with alignment on the mean is very undesirable for Cauchy distribution. The deficiency perhaps lies in the choice of arithmetic mean as our "norm" of alignment, rather than in the joint ranking procedure.

The median is a more stable measure of central tendency for the Cauchy distribution than the arithmetic mean and may be preferable. But the investigations of these questions seem rather complex and will not be taken up here.

II. Partially Balanced Incomplete Block Design

In this case, the corresponding separate ranking statistic

is given by

$$(4.2.9) \quad W_n^* = \{12(k-1)/nm(m-1)\} \sum_{j=1}^k \{R_j^* - \frac{r}{2}(m+1)\}^2$$

where R_j^* = sum of the ranks under the j^{th} treatment. Its large sample behaviour is given by the following theorem.

Theorem H. Assuming for each fixed n , the truth of the hypothesis K_n , if (i) $F(x)$ possesses a continuous derivative a.e. and (ii) there exists a function $b(x)$ such that $[F(x+\theta) - F(x)]/\theta \leq b(x)$ where

$$\int_{-\infty}^{\infty} b(x) dF(x) < \infty, \text{ then the statistic } W_n^* \text{ given by (4.2.9), converges,}$$

as $n \rightarrow \infty$, to $\chi_{k-1}^2(\Delta_{W^*}')$, where

$$(4.2.10) \quad \Delta_{W^*}' = \{12m(m-1)(k-1)/(m+1)k^2\} \left[\int_{-\infty}^{\infty} \{F'(x)\}^2 dx \right]^2 \sum_{j=1}^k \left\{ \theta_j - \frac{k}{m(m-1)} \cdot \sum_{j' \neq j=1}^k p_{jj'} \theta_{j'} \right\}^2$$

Proof: There are only a finite number $\binom{k}{m}$ ways of choosing m treatments from k and the vectors of observed ranks containing the same sets of treatments are independently identically distributed. The conclusion of the theorem now follows from the central limit theorem for random vectors.

From (4.2.10) and (3.3.3), taking the ratio of the non-centrality parameters, the efficiency expression is seen to agree with that given by (4.2.3). So the comments of the earlier section apply verbatim to this case.

The asymptotic efficiency relative to the F -test, however, depends upon the θ 's and thus does not convey much meaning.

III. "Balanced" Incomplete Blocks with p Observations per Cell. (For the meaning of "balanced" see section 3.4)

In this case the separate ranking statistic can be written in the form

$$(4.2.11) \quad W_n^* = \{ 12/p \cdot kp^2(mp+1) \} \sum_{j=1}^k \{ R_j^* - \frac{rp(mp+1)}{2} \}^2$$

If the conditions of Theorem G are satisfied, the statistic W_n^* tends, as $n \rightarrow \infty$, to a $\chi^2_{k-1}(\Delta''_{W^*})$ variable, where

$$(4.2.12) \quad \Delta''_{W^*} = \{ 12p^2m(m-1)/(k-1)(mp+1) \} \left[\int_{-\infty}^{\infty} \{F'(x)\}^2 dx \right]^2 \sum_{j=1}^k (\theta_j - \bar{\theta})^2$$

This result can be proved in exactly the same way as Theorem G.

The classical F-statistic is distributed in the limit, as $n \rightarrow \infty$, as $\chi^2_{k-1}(\Delta''_F)$, where

$$(4.2.13) \quad \Delta''_F = [p(m-1) \sum_{j=1}^k (\theta_j - \bar{\theta})^2] / [(k-1)\sigma^2]$$

if the variance σ^2 for the parent distribution exists.

From (4.2.12), (4.2.13) and (3.4.3), taking the ratio of the non-centrality parameters, we get

$$(4.2.14) \quad e_{W, W^*} = [(m-1)(mp+1)(kp-1)/4mp(mp-1)(k-1)(1-3\lambda)] \cdot \left[\int_{-\infty}^{\infty} \{G'(x)\}^2 dx / \int_{-\infty}^{\infty} \{F'(x)\}^2 dx \right]^2$$

$$(4.2.15) \quad e_{W, F} = [3\sigma^2(kp-1)(m-1)/(mp-1)(k-1)(1-3\lambda)] \left[\int_{-\infty}^{\infty} \{G'(x)\}^2 dx \right]^2$$

These expressions are again of the same type as (4.2.3) and (4.2.4).

It is interesting to note that if m and k are kept fixed and p allowed to tend to infinity, then using results of Theorem 4.1

$$(4.2.16) \quad \lim_{p \rightarrow \infty} e_{w, w^*} = \frac{k(m-1)}{m(k-1)} \leq 1$$

if the parent distribution has a finite expectation. We also note that the efficiency expression $e_{w, F}$ of (4.2.15), differs from that of (4.2.4) by a constant factor $\{(kp-1)(m-1)/(mp-1)(k-1)\}$ which is always less than unity and tends to $\{k(m-1)/m(k-1)\}$ as p tends to infinity.

Remark. The above "balanced" incomplete block design with p observations per cell can be treated as a partially balanced incomplete block with two associate classes where $\rho_{jj'} = r$ or ρ . We have the additional information that the treatment effects for plots occurring in the same cell are equal. However it follows from (4.2.16) that this additional information regarding θ_j 's contributes more towards the efficiency of the separate ranking than the joint ranking procedure. Accordingly, comparing the results of this section with those of the complete case (Mehra-Sarangi [14]), it seems that joint ranking procedure (taking account of interblock comparisons) would be preferable to separate ranking procedure only if the number of observations in the different cells are of the same order.

We, however, wish to remark that the limiting value of e_{w, w^*} , as p tends to infinity, will be rather close to unity if the value of m is close to k .

4.3 Joint Ranking Within Similar Blocks in a Special Case

In this section, two blocks are called similar if they contain the same set of treatments. Here, we consider a method of ranking in which all aligned observations within each set of similar blocks are ranked jointly together. Different similar sets are ranked

separately. A test statistic can be proposed by finding the limiting distribution of $(R_1, R_2, \dots, R_k)$ where R_j refers to the sum of the ranks under the j^{th} treatment. We specially consider a balanced incomplete block design with $n=Nd$ blocks containing d sets, each of N similar blocks. Suppose d is kept fixed and n tends to infinity with N . In the notations of sec. 3.2, we prove the following theorem.

Theorem 4.2. Suppose the hypothesis K_n be true and the conditions (i), (ii) and (iii) of Theorem 3.1 be satisfied. (See p.18). Then the statistic V_n , given by

$$(4.3.1) \quad V_n = \{3d^2(k-1)/m^2(m-1)(1-3\lambda)n^3\} \sum_{j=1}^k \{R_j - \frac{r}{2} (Nm+1)\}^2$$

converges in distribution, as $n \rightarrow \infty$, to $\chi^2_{k-1}(\Delta_w)$ where

$$(4.3.2) \quad \Delta_w = [3(m-1)/(1-3\lambda)(k-1)] \left[\int_{-\infty}^{\infty} \{G'(u)\}^2 du \right]^2 \sum_{j=1}^k (\theta_j - \bar{\theta})^2$$

is the same as given by (3.2.3).

Proof: Since the ranking involves d sets, each containing m out of the k treatments to be compared; each R_j can be regarded as the sum of (md/k) independent r.v 's. Using results of Lemma 3.1 (Mehta-Sarangi [14]), it is verified by direct computation that if

$$(4.3.3) \quad T_{nj} = [d\sqrt{3(k-1)} \{R_j - \frac{r}{2} (Nm+1)\}] / mn^{3/2} \sqrt{(m-1)(1-3\lambda)}$$

$$(4.3.4) \quad \mu_j = - \sqrt{3(m-1)/\{(k-1)(1-3\lambda)\}} \left[\int_{-\infty}^{\infty} \{G'(u)\}^2 du \right] (\theta_j - \bar{\theta})$$

then $\underline{T}_n = (T_{n1}, T_{n2}, \dots, T_{nk})$ has a k -variate normal distribution with mean vector $\underline{\mu} = (\mu_1, \mu_2, \dots, \mu_k)$ and variance-covariance matrix $\underline{\Lambda} = ||\delta_{jj} - \frac{1}{k}||$. The rest of the theorem follows from here, using

Lemma 3.2 (Mehra-Sarangi [14]), just as Theorem 3.1 followed from Lemmas 3.1 and 3.2.

It is seen from (4.3.2) and (3.2.3) that in our special case, joint ranking within similar blocks is asymptotically as efficient as the overall joint ranking procedure. This result is striking and suggests investigation of the fact: does the asymptotic relative efficiency of the joint ranking procedure within similar blocks compared to the overall joint ranking procedure tend, as the sequence of alternatives approach the null hypothesis, to unity from above or from below? It may also be of interest to try this method of alignment on other experimental designs. But these questions will not be considered here any further.

CHAPTER V

CONCLUDING REMARKS

5.1 A Comparison With Competing Test Procedures

It is a problem of vital importance to compare the present test procedure with other competing test procedures. This question was partially answered in the last chapter by comparison with separate ranking procedure and F -test. In the present work the efficiency expressions $e_{w,F}(F)$ and $e_{w,w^*}(F)$ were evaluated only for the case of a normal parent distribution. The problem of evaluating them for other parent distributions e.g. the uniform, logistic, (double) exponential, Cauchy and others needs additional investigation. These values are available for $m=2$ and when m tends to infinity (Mehra-Sarangi [14]). The value of $e_{w,F}$ can be arbitrarily high and even infinite as in the case of a parent Cauchy distribution. It may be possible to find a lower bound for $e_{w,F}$, comparable to the one obtained by Hodges-Lehmann in [7], for the efficiency of the Wilcoxon (Kruskal-Wallis) test relative to the t -test (F -test). The Lemma 5.1 below provides us with a lower bound for $e_{w,F}(F)$ for the case when F is a symmetric distribution.

Lemma 5.1 If the parent distribution $F(x)$ is symmetrical, then
 $\lambda \geq 1/6$ and $e_{w,F}(F) \geq 0.432 m/m-1$.

Proof. If the parent distribution is symmetrical, then both $G(x)$ and $H(x,y)$ are symmetrical about the origin and the line $y = x$, thus giving

$$(5.1.1) \quad G(x) + G(-x) = 1$$

$$(5.1.2) \quad H(-x, -y) = 1 - G(x) - G(y) + H(x,y)$$

Hence,

$$\begin{aligned}
 \lambda &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x)G(y) \, dH(x,y) \\
 &= \int_0^{\infty} \int_0^{\infty} \{G(x)G(y)+G(-x)G(-y)\} \, dH(x,y) + \int_0^{\infty} \int_0^{\infty} \{G(x)G(-y)+G(-x)G(y)\}dH(-x,y) \\
 &= \int_0^{\infty} \int_0^{\infty} \{1-G(x)-G(y)+2G(x)G(y)\}dH(x,y) + \int_0^{\infty} \int_0^{\infty} \{G(x)+G(y)-2G(x)G(y)\}dH(-x,y) \\
 &= \int_0^{\infty} \int_0^{\infty} \{1 - 2G(x) - 2G(y) + 4G(x)G(y)\}dH(x,y) \\
 &+ \int_0^{\infty} \int_{-\infty}^{\infty} \{G(x) - G^2(x)\}dH(x,y) + \int_{-\infty}^{\infty} \int_0^{\infty} \{G(y) - G^2(y)\}dH(x,y) \\
 &+ \int_0^{\infty} \int_0^{\infty} \{G(x) - G(y)\}^2dH(x,y) + \int_0^{\infty} \int_0^{\infty} \{G(x) - G(y)\}^2dH(-x,y) \\
 &= \frac{1}{6} + \int_0^{\infty} \int_0^{\infty} \{2G(x)-1\} \{2G(y)-1\} \, dH(x,y) + \int_0^{\infty} \int_0^{\infty} \{G(x) - G(y)\}^2dH(x,y) \\
 &+ \int_0^{\infty} \int_0^{\infty} \{G(x) - G(y)\}^2dH(-x,y) \geq \frac{1}{6}
 \end{aligned}$$

since each term in the above expression is non negative. In the above simplifications the first integral sign contains the limits for x and the second contains the limits for y.

Again,

$$\begin{aligned}
 e_{w,F} &= \left[3\sigma^2/(1-3\lambda) \right] \left[\int_{-\infty}^{\infty} \{G'(x)\}^2 dx \right]^2 \\
 &= 12\sigma^2(m-1) \left\{ \int_{-\infty}^{\infty} \left(G'(x) \right)^2 dx \right\}^2 / m \cdot \{m/(m-1)\} \{1/4(1-3\lambda)\} \\
 &\geq 0.432 \, m/(m-1) \, .
 \end{aligned}$$

since the variance of the r.v. with c.d.f. $G(x)$ is $(m-1)\sigma^2/m$ and thus by the results proved in Hodges-Lehmann [7], the quantity within the first square brackets is not less than 0.864; and $\lambda \geq 1/6$. The result trivially holds if the parent distribution does not have a finite variance. This proves Lemma 5.1. It may be possible to improve upon the lower bound (as a function of m) of λ by finding lower bounds of the non-negative terms that were ignored in proving $\lambda \geq 1/6$.

For the case of normal parent distribution, the conditional test considered here is slightly less efficient than the test proposed by Lehmann in [12] but is more efficient than his test proposed in [11]. But it is not at all clear how the present test stands in relation to these for other forms of the parent distribution. However this conditional test has the important advantage of providing exact levels of significance for all sample sizes, whereas the tests proposed by Lehmann in [11] and [12] are only asymptotically distribution-free.

The first of these is the fact that the system is not a simple one. It is a complex system, and the behavior of the system is not predictable. The second is that the system is not a simple one. It is a complex system, and the behavior of the system is not predictable. The third is that the system is not a simple one. It is a complex system, and the behavior of the system is not predictable. The fourth is that the system is not a simple one. It is a complex system, and the behavior of the system is not predictable. The fifth is that the system is not a simple one. It is a complex system, and the behavior of the system is not predictable. The sixth is that the system is not a simple one. It is a complex system, and the behavior of the system is not predictable. The seventh is that the system is not a simple one. It is a complex system, and the behavior of the system is not predictable. The eighth is that the system is not a simple one. It is a complex system, and the behavior of the system is not predictable. The ninth is that the system is not a simple one. It is a complex system, and the behavior of the system is not predictable. The tenth is that the system is not a simple one. It is a complex system, and the behavior of the system is not predictable.

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